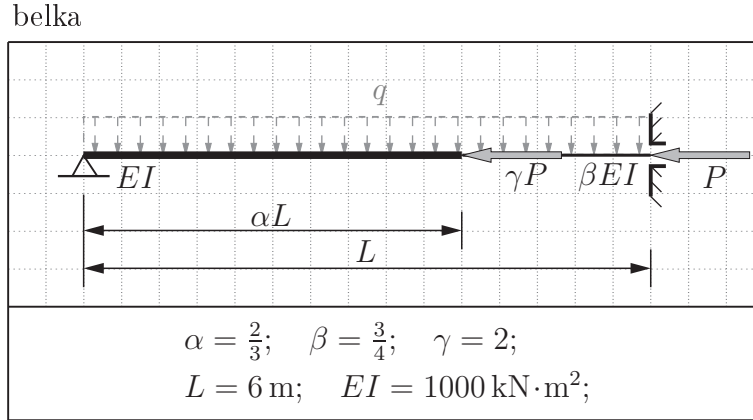


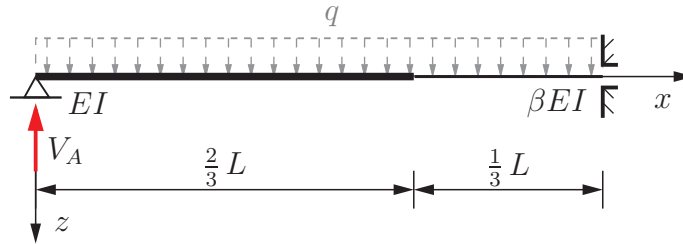
Przykład do zadania 6.

Dla pręta pokazanego na rysunku wyznaczyć krytyczną wartość siły P oraz współczynnik wyboczeniowy μ . Wykorzystać energetyczne kryterium Timoshenki przyjmując jako postulowaną postać wyboczenia linię ugięcia belki wyznaczoną dla zadanego obciążenia rozłożonego q .



1. Postać wyboczenia

1.1. Równanie różniczkowe osi odkształconej



I) $0 \leq x \leq \frac{2}{3} L$

$$\begin{aligned}
 M_y(x) &= V_A x - \frac{1}{2} q x^2 \\
 EI w''(x) &= \frac{1}{2} q x^2 - V_A x \\
 EI w'(x) &= \frac{1}{6} q x^3 - \frac{1}{2} V_A x^2 + C_1 \\
 EI w(x) &= \frac{1}{24} q x^4 - \frac{1}{6} V_A x^3 + C_1 x + D_1
 \end{aligned}$$

II) $\frac{2}{3} L \leq x \leq L$

$$\begin{aligned}
 M_y(x) &= V_A x - \frac{1}{2} q x^2 \\
 \frac{3}{4} EI w''(x) &= \frac{1}{2} q x^2 - V_A x \\
 \frac{3}{4} EI w'(x) &= \frac{1}{6} q x^3 - \frac{1}{2} V_A x^2 + C_2 \\
 \frac{3}{4} EI w(x) &= \frac{1}{24} q x^4 - \frac{1}{6} V_A x^3 + C_2 x + D_2
 \end{aligned}$$

1.2. Wyznaczenie stałych z warunków brzegowych

$$\left. \begin{aligned}
 (1) \quad w_I(0) &= 0 & \Rightarrow D_1 &= 0 \\
 (2) \quad w'_I\left(\frac{2}{3}L\right) &= w'_{II}\left(\frac{2}{3}L\right) & \Rightarrow C_1 - \frac{4}{3} C_2 + \frac{2}{27} V_A L &= \frac{4}{243} q L^3 \\
 (3) \quad w_I\left(\frac{2}{3}L\right) &= w_{II}\left(\frac{2}{3}L\right) & \Rightarrow \frac{2}{3} C_1 L - \frac{8}{9} C_2 L - \frac{4}{3} D_2 + \frac{4}{243} V_A L &= \frac{2}{729} q L^4 \\
 (4) \quad w'_{II}(L) &= 0 & \Rightarrow C_2 - \frac{1}{2} V_A L^2 &= -\frac{1}{6} q L^3 \\
 (5) \quad w_{II}(L) &= 0 & \Rightarrow C_2 L + D_2 - \frac{1}{6} V_A L^3 &= -\frac{1}{24} q L^4
 \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} C_1 &= \frac{136}{6075} q L^3 \\ C_2 &= \frac{31}{1200} q L^3 \\ D_1 &= 0 \\ D_2 &= -\frac{1}{300} q L^4 \\ V_A &= \frac{77}{200} q L \end{aligned} \right.$$

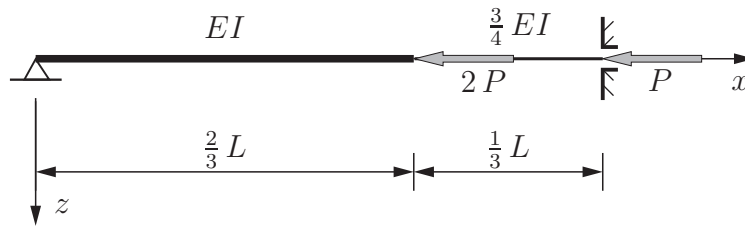
1.3. Rysunek postaci odkształconej belki



2. Obciążenie krytyczne

2.1. Kryterium energetyczne Timoshenki

$$\int_L EI (w'')^2 dx + \int_L N (w')^2 dx = 0$$



$$\left. \begin{aligned} w'_I(x) &= \frac{1}{EI} \left(\frac{1}{6} qx^3 - \frac{77}{400} qLx^2 + \frac{136}{6075} qL^3 \right) \\ w''_I(x) &= \frac{1}{EI} \left(\frac{1}{2} qx^2 - \frac{77}{200} qLx \right) \\ N_I(x) &= -3P \end{aligned} \right| \begin{aligned} w'_{II}(x) &= \frac{4}{3EI} \left(\frac{1}{6} qx^3 - \frac{77}{400} qLx^2 + \frac{31}{1200} qL^3 \right) \\ w''_{II}(x) &= \frac{4}{3EI} \left(\frac{1}{2} qx^2 - \frac{77}{200} qLx \right) \\ N_{II}(x) &= -P \end{aligned}$$

$$\int_0^{\frac{2}{3}L} EI (w''_I)^2 dx + \int_{\frac{2}{3}L}^L \frac{3}{4} EI (w''_{II})^2 dx - P \int_0^{\frac{2}{3}L} 3 (w'_I)^2 dx - P \int_{\frac{2}{3}L}^L (w'_{II})^2 dx = 0$$

2.2. Wartość krytyczna siły P

$$P_{kr} = \frac{I_1 + I_2}{I_3 + I_4}$$

$$\begin{aligned} I_1 &= \int_0^{\frac{2}{3}L} EI (w''_I)^2 dx = \frac{q^2}{EI} \int_0^{\frac{2}{3}L} \left(\frac{1}{2} x^2 - \frac{77}{200} Lx \right)^2 dx = \frac{q^2}{EI} \int_0^{\frac{2}{3}L} \left(\frac{1}{4} x^4 - \frac{77}{200} Lx^3 + \frac{77^2}{200^2} L^2 x^2 \right) dx = \\ &= \frac{q^2}{EI} \left(\frac{1}{20} x^5 - \frac{77}{800} Lx^4 + \frac{5929}{120000} L^2 x^3 \right) \Big|_0^{\frac{2}{3}L} = \frac{2687}{1215000} \frac{q^2 L^5}{EI} = 2,212 \cdot 10^{-3} \frac{q^2 L^5}{EI} \end{aligned}$$

$$I_2 = \int_{\frac{2}{3}L}^L \frac{3}{4} EI (w''_{II})^2 dx = \frac{4q^2}{3EI} \int_{\frac{2}{3}L}^L \left(\frac{1}{2} x^2 - \frac{77}{200} Lx \right)^2 dx =$$

$$= \frac{4q^2}{3EI} \left(\frac{1}{20}x^5 - \frac{77}{800}Lx^4 + \frac{5929}{120000}L^2x^3 \right) \Big|_{\frac{2}{3}L}^L = \frac{9203}{7290000} \frac{q^2L^5}{EI} = 1,262 \cdot 10^{-3} \frac{q^2L^5}{EI}$$

$$\begin{aligned} I_3 &= \int_0^{\frac{2}{3}L} 3(w'_I)^2 dx = \frac{3q^2}{EI^2} \int_0^{\frac{2}{3}L} \left(\frac{1}{6}x^3 - \frac{77}{400}Lx^2 + \frac{136}{6075}L^3 \right)^2 dx = \\ &= \frac{3q^2}{EI^2} \int_0^{\frac{2}{3}L} \left(\frac{1}{36}x^6 - \frac{77}{1200}Lx^5 + \frac{5929}{160000}L^2x^4 + \frac{136}{18225}L^3x^3 - \frac{1309}{151875}L^4x^2 + \frac{18496}{36905625}L^6 \right) dx = \\ &= \frac{3q^2}{EI^2} \left(\frac{1}{252}x^7 - \frac{77}{7200}Lx^6 + \frac{5929}{800000}L^2x^5 + \frac{34}{18225}L^3x^4 - \frac{1309}{455625}L^4x^3 + \frac{18496}{36905625}L^6x \right) \Big|_0^{\frac{2}{3}L} = \\ &= 3,621 \cdot 10^{-4} \frac{q^2L^7}{EI^2} \end{aligned}$$

$$\begin{aligned} I_4 &= \int_{\frac{2}{3}L}^L (w'_{II})^2 dx = \frac{16q^9}{EI^2} \int_{\frac{2}{3}L}^L \left(\frac{1}{6}x^3 - \frac{77}{400}Lx^2 + \frac{31}{1200}L^3 \right)^2 dx = \\ &= \frac{16q^2}{9EI^2} \int_{\frac{2}{3}L}^L \left(\frac{1}{36}x^6 - \frac{77}{1200}Lx^5 + \frac{5929}{160000}L^2x^4 + \frac{31}{3600}L^3x^3 - \frac{2387}{240000}L^4x^2 + \frac{961}{1440000}L^6 \right) dx = \\ &= \frac{16q^2}{9EI^2} \left(\frac{1}{252}x^7 - \frac{77}{7200}Lx^6 + \frac{5929}{800000}L^2x^5 + \frac{31}{14400}L^3x^4 - \frac{2387}{720000}L^4x^3 + \frac{961}{1440000}L^6x \right) \Big|_{\frac{2}{3}L}^L = \\ &= 5,820 \cdot 10^{-5} \frac{q^2L^7}{EI^2} \end{aligned}$$

$$P_{kr} = \frac{(2,212 \cdot 10^{-3} + 1,262 \cdot 10^{-3}) \frac{q^2L^5}{EI}}{(3,621 \cdot 10^{-4} + 5,820 \cdot 10^{-5}) \frac{q^2L^7}{EI^2}} = 8,265 \frac{EI}{L^2} = 8,265 \frac{1000}{6^2} = 229,6 \text{ kN}$$

$$\mu = \sqrt{\frac{\pi^2 EI}{P_{kr} L^2}} = \sqrt{\frac{\pi^2}{8,265}} = 1,093$$